4/13/18

"In order to understand the value in a shortcut, one must have taken the long road first."
-Anonymous

HW: "Exponential Modeling Day 2" homework section

AIM: Exponential Modeling Cont.

Warm Up:

Percents combine in strange ways that don't seem to make sense at first. It would seem that if a population grows by 5% per year for 10 years, then it should grow in total by 50% over a decade. But this isn't true. Start with a population of 100. If it grows at 5% per year for 10 years, what is its population after 10 years? What percent growth does this represent?

$A = P(1+r)^t$

$A = 100(1+.05)^{10}$

$A = 162.889$

$(1.05)^{10}$

1.62889

$.62889$ ~~%~~

62.889 %

Total % Growth

APPLICATIONS

1. If \$130 is invested in a savings account that earns 4% interest per year, which of the following is closest to the amount in the account at the end of 10 years?

(1) \$218 (3) \$168

(2) \$192 (4) \$324

$\boxed{130(1.04)^{10} = 192.431... \approx \$192}$

$\boxed{(2)}$

2. A population of 50 fruit flies is increasing at a rate of 6% per day. Which of the following is closest to the number of days it will take for the fruit fly population to double?

(1) 18 (3) 12

(2) 6 (4) 28

$50(1.06)^t = 100$

Use tables on your calculator.

$50(1.06)^{12} \approx 100$

$\boxed{(3)}$

3. If a radioactive substance is quickly decaying at a rate of 13% per hour approximately how much of a 200 pound sample remains after one day? 24 hrs

(1) 7.1 pounds (3) 25.6 pounds

(2) 2.3 pounds (4) 15.6 pounds

$200(1-.13)^{24} = 200(.87)^{24}$
$= 7.071... \approx 7.1$

$\boxed{(1)}$

4. A population of llamas stranded on a desert island is decreasing due to a food shortage by 6% per year. If the population of llamas started out at 350, how many are left on the island 10 years later?

(1) 257 (3) 102

(2) 58 (4) 189

$350(1-.06)^{10} = 350(.94)^{10}$
$= 188.515... \approx 189$

$\boxed{(4)}$

5. Which of the following equations would model a population with an initial size of 625 that is growing at an annual rate of 8.5%?

(1) $P = 625(8.5)^t$ (3) $P = 1.085^t + 625$

(2) $P = 625(1.085)^t$ (4) $P = 8.5t^2 + 625$

$P = 625(1+.085)^t$
$P = 625(1.085)^t$

$\boxed{(2)}$

6. The acceleration of an object falling through the air will decrease at a rate of 15% per second due to air resistance. If the initial acceleration due to gravity is 9.8 meters per second per second, which of the following equations best models the acceleration *t* seconds after the object begins falling?

(1) $a = 15 - 9.8t^2$ (3) $a = 9.8(1.15)^t$

(2) $a = \dfrac{9.8}{15t}$ (4) $a = 9.8(0.85)^t$

$a = 9.8(1-.15)^t$
$a = 9.8(0.85)^t$

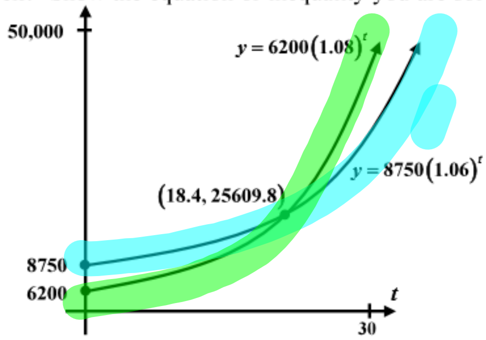
$\boxed{(4)}$

7. Red Hook has a population of 6,200 people and is growing at a rate of 8% per year. Rhinebeck has a population of 8,750 and is growing at a rate of 6% per year. In how many years, to the nearest year, will Red Hook have a greater population than Rhinebeck? Show the equation or inequality you are solving and solve it graphically.

$$6200(1.08)^t > 8750(1.06)^t$$

Solved graphically ⇒

$t = 18$ **years**



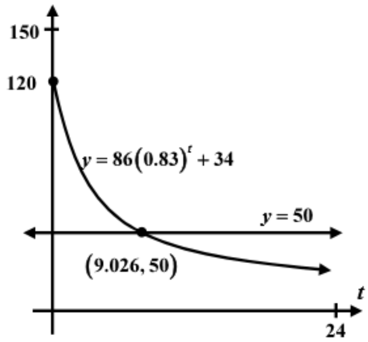
8. A warm glass of water, initially at 120 degrees Fahrenheit, is placed in a refrigerator at 34 degrees Fahrenheit and its temperature is seen to decrease according to the exponential function

$$T(h) = 86(0.83)^h + 34$$

(a) Verify that the temperature starts at 120 degrees Fahrenheit by evaluating $T(0)$.

$$T(0) = 86(.83)^0 + 34$$
$$= 86(1) + 34 = 120$$

(b) Using your calculator, sketch a graph of T below for all values of h on the interval $0 \le h \le 24$. Be sure to label your y-axis and y-intercept.



(c) After how many hours will the temperature be at 50 degrees Fahrenheit? State your answer to the nearest *hundredth* of an hour. Illustrate your answer on the graph you drew in (b).

$t = \log_{.83} \frac{16}{86}$

$$86(0.83)^t + 34 = 50$$
$$-34 \quad -34$$
~~$t = 9.03$ hrs~~

$86(.83)^t = 16$

$.83^t = \frac{16}{86}$

REASONING

9. Percents combine in strange ways that don't make sense at first. It would seem that if a population grows by 5% per year for 10 years, then it should grow in total by 50% over a decade. But this isn't true. Start with a population of 100. If it grows at 5% per year for 10 years, what is its population after 10 years? What percent growth does this represent?

$$100(1.05)^{10} = 162.8894 \approx 163$$

⇒

The population actually grew by around 63% instead of the 50% that we would naturally think.

increase   named rate

***Exercise* #1:** A person invests $500 in an account that earns a **nominal yearly interest rate** of 4%.

(a) How much would this investment be worth in 10 years if the **compounding frequency** was once per year? Show the calculation you use.

\# of times per year

$A = P(1+r)^t$

$A = 500(1+.04)^{10}$

$= \$740.12$

(b) If, on the other hand, the interest was applied four times per year (known as quarterly compounding), why would it not make sense to multiply by 1.04 each quarter?

No b/c 4% is the yearly rate

(c) If you were told that an investment earned 4% per year, how much would you assume was earned per quarter? Why?

$\frac{4\%}{4} = 1\%$

(d) Using your answer from part (c), calculate how much the investment would be worth after 10 years of quarterly compounding? Show your calculation.

4 times each year

$A = P(1+r)^t$

$= 500(1+.01)^{40}$

$= \$744.43$

So, the pattern is fairly straightforward. For a **shorter compounding period**, we get to **apply the interest more often**, but at a **lower rate**.

***Exercise* #2:** How much would $1000 invested at a nominal 2% yearly rate, compounded monthly, be worth in 20 years? Show the calculations that lead to your answer.

12 times

(1) $1485.95

(2) $1491.33

(3) $1033.87

(4) $1045.32

$A = P(1+r)^t$

$\frac{.02}{12} =$

$A = 1000\left(1+\frac{.02}{12}\right)^{20 \cdot 12}$

$= \$1491.33$

This pattern is formalized in a classic formula from economics that we will look at in the next exercise.

***Exercise* #3:** For an investment with the following parameters, write a formula for the amount the investment is worth, A, after t-years.

P = amount initially invested

r = nominal yearly rate

n = number of compounds per year

$$\boxed{A(t) = P\left(1+\frac{r}{n}\right)^{nt}}$$

actual rate

***Exercise* #4:** An investment with a nominal rate of 5% is compounded at different frequencies. Give the **effective** yearly rate, accurate to two decimal places, for each of the following compounding frequencies. Show your calculation.

(a) Quarterly

$$\left(1+\frac{.05}{4}\right)^{4\cdot 1}$$

$$1\boxed{.0509}$$

5.09%

(b) Monthly

$$\left(1+\frac{.05}{12}\right)^{12\cdot 1}$$

$$1\boxed{.05116}$$

5.12%

(c) Daily

$$\left(1+\frac{.05}{365}\right)^{365\cdot 1}$$

1.05126

5.13%

We could compound at smaller and smaller frequency intervals, eventually compounding all moments of time. In our formula from *Exercise* #3, we would be letting n approach infinity. Interestingly enough, this gives rise to **continuous compounding** and the use of the natural base **e** in the famous **continuous compound interest formula.**

CONTINUOUS COMPOUND INTEREST

For an initial principal, P, compounded continuously at a nominal yearly rate of r, the investment would be worth an amount A given by:

$$A(t) = Pe^{rt}$$

***Exercise* #5:** A person invests \$350 in a bank account that promises a nominal rate of 2% continuously compounded.

(a) Write an equation for the amount this investment would be worth after t-years.

$$A = 350e^{.02t}$$

(b) How much would the investment be worth after 20 years? $= t$

$$A = 350e^{.02(20)}$$

$$\boxed{A = \$522.14}$$

(c) Algebraically determine the time it will take for the investment to reach \$400. Round to the nearest tenth of a year.

A

$$\frac{400}{350} = \frac{350e^{.02t}}{350}$$

$$\frac{400}{350} = e^{.02t}$$

rewrite using logs

$$\frac{.02t}{.02} = \frac{\ln \frac{400}{350}}{.02}$$

$$\boxed{t \approx 6.7 \text{ years}}$$

(d) What is the effective annual rate for this investment? Round to the nearest hundredth of a percent.

$$A = Pe^{rt}$$

$$e^{.02(1)} = \underline{1.0202}$$

$$\boxed{2.02\%}$$